AN EXTREMAL PROBLEM FOR POSITIVE DEFINITE MATRICES

BY

T. W. ANDERSON and I. OLKIN

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OFFICE OF NAVAL RESEARCH

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Abstract

A problem studied by Flanders (1975) is to minimize the function $f(R) = \operatorname{tr}(SR + TR^{-1})$ over the set of positive definite matrices R, where S and T are positive semi-definite matrices. Alternative proofs that may have some intrinsic interest are provided. The proofs explicitly yield the infimum of $f(R)$. One proof is based on a convexity argument and the other on a sequence of reductions to a univariate problem.

1. Introduction.

Flanders (1975) studied a matrix problem that arose in electric circuit theory. Let $z_1, \ldots, z_m$, $w_1, \ldots, w_m$ be complex column vectors of length n and consider the real-valued function

$$f(R) = z_1^* R z_1 + \cdots + z_m^* R z_m + w_1^* R^{-1} w_1 + \cdots + w_m^* R^{-1} w_m \,, \tag{1}$$

where $R > 0$ denotes an $n \times n$ positive definite Hermitian matrix. The problem is to minimize $f(R)$ over the set of positive definite R.

If we set

$$Z = (z_1, \ldots, z_m) \,, \qquad W = (w_1, \ldots, w_m),$$

$$S = ZZ^* \,, \qquad T = WW^* \,,$$

then (1) becomes

$$(2) \qquad f(R) = \operatorname{tr} R(z_1 z_1^* + \cdots + z_m z_m^*) + \operatorname{tr} R^{-1}(w_1 w_1^* + \cdots + w_m w_m^*)$$

$$= \operatorname{tr} R\, ZZ^* + \operatorname{tr} R^{-1}\, WW^*$$

$$= \operatorname{tr} R\, S + \operatorname{tr} R^{-1}\, T \,,$$

where S and T are positive semi-definite matrices of order n. We shall write $A \geq 0$ and $A > 0$ to denote that A is positive semi-definite and positive definite, respectively.

The result obtained by Flanders (1975) is as follows:

Theorem 1. If $f(R)$ is defined on the set of positive definite matrices by (1) or (2), and if $A = Z^*W$, then

$$\text{(i)} \quad \inf_{R > 0} f(R) = 2 \operatorname{tr} (AA^*)^{1/2} = 2 \operatorname{tr}(A^*A)^{1/2} \,,$$

(ii) $f(R_0) = 2 \operatorname{tr}(AA^*)^{1/2}$ for some $R_0 > 0$ if and only if $\operatorname{rank}(Z) = \operatorname{rank}(W) = \operatorname{rank}(A)$.

Flanders first proves that

$$f(R) \geq 2 \operatorname{tr}(AA^*)^{1/2} \,,$$

and then discusses the approach to equality. However, the matrix R that achieves equality (when the condition of (ii) is satisfied) is not exhibited in any simple manner. We now provide alternative somewhat simpler proofs of (i) that may have some intrinsic interest.

2. Matrices are of full rank.

First Alternative Proof. This proof is based on the fact that $f(R)$ is a convex function of R. The function $\operatorname{tr} RS$ is linear in R, and $\operatorname{tr} R^{-1}T$ is convex in R, i.e.,

$$\operatorname{tr}(\alpha R_1 + (1-\alpha)R_2)^{-1} T \leq \alpha \operatorname{tr} R_1^{-1}T + (1-\alpha) \operatorname{tr} R_2^{-1}T, \quad 0 \leq \alpha \leq 1 ,$$

for R_1 and R_2 positive definite. The inequality is strict unless $R_1 = R_2$ or $\alpha = 0$ or 1. Consequently, $f(R)$ is (strictly) convex, and we need to minimize a convex function over a convex set. Since $f(R) \to \infty$ as $R \to 0$ or as $R \to \infty$, the minimum is achieved at an interior point, namely where $df(R)/dR = 0$. But

$$df = \operatorname{tr} S(dR) - \operatorname{tr} R^{-1} (dR)R^{-1}T = \operatorname{tr} dR(S-R^{-1}TR^{-1}) ;$$

so that there is an interior point $\tilde{R}$ satisfying $S-\tilde{R}^{-1}T\tilde{R}^{-1} = 0$, or equivalently,

(3) $$\tilde{R}\, S\, \tilde{R} = T ,$$

that is the minimizer of $f(R)$. Note that $\operatorname{tr} \tilde{R}S = \operatorname{tr} \tilde{R}^{-1}T$, so that

$$f(\tilde{R}) = 2 \operatorname{tr} \tilde{R}S .$$

Furthermore, from (3),

$$S^{1/2}\tilde{R}S\tilde{R}S^{1/2} = (S^{1/2}\tilde{R}S^{1/2})^2 = S^{1/2}TS^{1/2} ;$$

so that

$$S^{1/2}\tilde{R}S^{1/2} = (S^{1/2}TS^{1/2})^{1/2} ,$$

and

$$f(\tilde{R}) = 2 \text{ tr } \tilde{R}S = 2 \text{ tr}(S^{1/2}TS^{1/2})^{1/2} .$$

Here we have used the positive definite square root. However, any square root, e.g., $S = LL^*$, $T = MM^*$ can be used, in which case the result is

$$f(\tilde{R}) = 2 \text{ tr}(L^*MM^*L)^{1/2} = 2 \text{ tr}(M^*LL^*M)^{1/2} .$$

Let $\lambda_i(A)$ denote the characteristic roots of the matrix A. Because

$$\begin{aligned} \text{tr}(L^*MM^*L)^{1/2} &= \Sigma\, \lambda_i[(L^*MM^*L)^{1/2}] = \Sigma[\lambda_i(L^*MM^*L)]^{1/2} \\ &= \Sigma[\lambda_i(MM^*LL^*)]^{1/2} = \Sigma[\lambda_i(TS)]^{1/2} , \\ &= \text{tr}(T^{1/2}ST^{1/2})^{1/2} = \text{tr}(S^{1/2}TS^{1/2})^{1/2} , \end{aligned}$$

we see that all square roots yield the same result.

Second Alternative Proof. This proof is based on a sequence of reductions until finally we obtain an extremal problem of distinct variables. There exists a nonsingular matrix H such that

$$T^{-1} = HH^* , \quad S = HD_d H^* ,$$

where D_d is a diagonal matrix with real elements $(d_1,\ldots,d_n)$ ordered $0 < d_1 \leq \cdots \leq d_n$, which are the roots of

$$0 = |S-yT^{-1}| = |T^{-1}| \cdot |ST-yI| = |T^{-1}| \cdot |S^{1/2}TS^{1/2}-yI| .$$

Then

$$\begin{aligned} f(R) &= \operatorname{tr} RHD_d H^* + \operatorname{tr} R^{-1}H^{*-1}H^{-1} \\ &= \operatorname{tr} H^*RHD_d + \operatorname{tr}(H^*RH)^{-1} \equiv \operatorname{tr} GD_d + \operatorname{tr} G^{-1} , \end{aligned}$$

where $G = H^*RH$. Let $G = QD_\lambda Q^*$, where Q is unitary and $D_\lambda = \operatorname{diag}(\lambda_1,\ldots,\lambda_n)$ is diagonal with positive diagonal elements ordered $0 < \lambda_n \leq \cdots \leq \lambda_1$. The function to be minimized (with respect to Q and D_λ) is

$$\operatorname{tr} QD_\lambda Q^* D_d + \operatorname{tr} D_\lambda^{-1} = \sum_{i,j=1}^{n} d_i \lambda_j |q_{ij}|^2 + \sum_{j=1}^{n} \lambda_j^{-1} . \tag{4}$$

By a theorem of von Neumann (1937) [see also Fan (1951)], the minimum of the first sum in (4) with respect to Q is $\Sigma_{j=1}^n d_j\lambda_j$, and the minimizing Q is I. Then the minimum of $\Sigma_{j=1}^{n} d_j\lambda_j + \Sigma_{j=1}^{n} \lambda_j^{-1}$ with respect to $\lambda_1,\ldots,\lambda_n$, is attained at $\lambda_j = d_j^{-1/2}$ and the minimized value of (4) is $2\,\Sigma_{j=1}^{n} d_j^{1/2}$.

<u>Remark</u>. If one or more of the d_j are 0, then the infimum cannot be attained.

The minimizing matrix R is

$$R = (H^*)^{-1}GH^{-1} = (H^*)^{-1}D_d^{-1/2}H^{-1} = (HD_d^{-1/2}H^*)^{-1} .$$

This matrix satisfies (3).

3. <u>Matrices not full of rank</u>.

In the case when rank(S) and/or rank(T) is less than n we show how to reduce the problem to a canonical form from which we may then invoke the result for full rank.

Note that $f(R) = \mathrm{tr}\, RS + \mathrm{tr}\, R^{-1}T$ is invariant with respect to the transformation

$$(R,S,T) \to (QRQ^*, Q^{*-1}SQ^{-1}, QTQ^*)$$

for any nonsingular matrix Q. Furthermore, the ranks of S, T and ST are invariant under this transformation. By judiciously choosing a matrix Q we can effect a simplification of the problem.

<u>Theorem 2</u>. If $S \geq 0$, $T \geq 0$, then there exists a nonsingular matrix V such that

$$(5)\qquad \tilde{T} = VTV^* = \begin{pmatrix} I_\tau & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{S} \equiv V^{*-1}SV^{-1} = \begin{pmatrix} D_d & 0 \\ 0 & M \end{pmatrix},$$

where $\tau = \text{rank}(T)$, $D_d = \text{diag}(d_1,\ldots,d_\tau)$, $\text{rank}(M) = \text{rank}(S) - \text{rank}(ST)$, and $\text{rank}(D_d) = \text{rank}(ST)$.

Suppose for the moment that Theorem 2 holds. Then

$$f(R) = \text{tr}\, RV^* \begin{pmatrix} D_d & 0 \\ 0 & M \end{pmatrix} V + \text{tr}\, R^{-1}V^{-1} \begin{pmatrix} I_\tau & 0 \\ 0 & 0 \end{pmatrix} V^{*-1}$$

$$= \text{tr}\, \tilde{R} \begin{pmatrix} D_d & 0 \\ 0 & M \end{pmatrix} + \text{tr}\, \tilde{R}^{-1} \begin{pmatrix} I_\tau & 0 \\ 0 & 0 \end{pmatrix},$$

where $\tilde{R} = VRV^*$. Minimization of $f(R)$ over $R > 0$ is equivalent to minimization over $\tilde{R} > 0$. Consequently,

$$f(R) = \text{tr}\, \tilde{R}_{11}D_d + \text{tr}\, \tilde{R}_{22}M + \text{tr}(\tilde{R}^{-1})_{11}$$

$$= \text{tr}\, \tilde{R}_{11}D_d + \text{tr}\, \tilde{R}_{22}M + \text{tr}(\tilde{R}_{11} - \tilde{R}_{12}\tilde{R}_{22}^{-1}\tilde{R}_{21})^{-1}.$$

Since $\text{tr}(\tilde{R}_{11} - \tilde{R}_{12}\tilde{R}_{22}^{-1}\tilde{R}_{21})^{-1} \geq \text{tr}\, \tilde{R}_{11}^{-1}$, $f(R)$ is minimized by taking $\tilde{R}_{12} = 0$. Then

$$\min_{\tilde{R}_{12},\tilde{R}_{11},\tilde{R}_{22}} f(R) = \min_{\tilde{R}_{11},\tilde{R}_{22}} [\text{tr}\, \tilde{R}_{11}D_d + \text{tr}\, \tilde{R}_{11}^{-1} + \text{tr}\, \tilde{R}_{22}M].$$

Three rank cases need to be considered.

(i) $\text{Rank}(T) > \text{rank}(ST)$. Then one or more of the d_j are 0, in which case the infimum is not attained. (See discussion leading to the remark in the second alternative proof.)

(ii) $\text{Rank}(S) > \text{rank}(ST)$. Then $M \neq 0$ and the infimum of 0 is not attained.

(iii) $\text{Rank}(T) = \text{rank}(S) = \text{rank}(ST)$. Then $M = 0$ and $\text{rank}(D_d) = \tau$, so that

$$\inf_{\tilde{R} > 0} f(\tilde{R}) = \inf_{\tilde{R}_{11} > 0} (\text{tr}\, \tilde{R}_{11} D_d + \text{tr}\, \tilde{R}_{11}^{-1})$$

and the problem has been reduced to the case of the one with full rank. This completes the proof of Theorem 1. $\|$

To prove Theorem 2 we use the following two lemmas.

<u>Lemma 1</u>. A nonsingular matrix

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}, \quad G_{11}: \tau \times \tau$$

satisfies

$$(6) \qquad G \begin{pmatrix} I_\tau & 0 \\ 0 & 0 \end{pmatrix} G^* = \begin{pmatrix} I_\tau & 0 \\ 0 & 0 \end{pmatrix}$$

if and only if $G_{21} = 0$, G_{11} is unitary, and G_{22} is nonsingular.

<u>Proof</u>. Multiply in (6) to obtain $G_{11}G_{11}^* = I_\tau$, $G_{21}G_{21}^* = 0$, from which the conclusion follows. $\|$

<u>Lemma 2</u>. Given $Q \geq 0$ there exists a G_{12} and a nonsingular G_{22} such that

$$Q_{11}G_{12} + Q_{12}G_{22} = 0 \, .$$

Proof. Q has the triangular decomposition

$$Q = \begin{pmatrix} T'_{11} & 0 \\ T'_{12} & T'_{22} \end{pmatrix} \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} = \begin{pmatrix} T'_{11}T_{11} & T'_{11}T_{12} \\ T'_{12}T_{11} & T'_{12}T_{12}+T'_{22}T_{22} \end{pmatrix} .$$

The rows of (T_{11},T_{12}) span a space of dimension less than or equal to τ. The $n-\tau$ columns of $\binom{G_{12}}{G_{22}}$ can be chosen to be orthogonal to the rows of (T_{11},T_{12}) and linearly independent (so that G_{22} is nonsingular). ||

Proof of Theorem 2. Let U be any nonsingular matrix such that

$$T = U \begin{pmatrix} I_\tau & 0 \\ 0 & 0 \end{pmatrix} U^* ,$$

and define

$$\hat{S} = U^*SU ,$$

Let G be any nonsingular matrix satisfying Lemma 2 and define

$$\tilde{S} = G^*\hat{S}G$$

$$= \begin{pmatrix} G^*_{11}\hat{S}_{11}G_{11} & G^*_{11}(\hat{S}_{11}G_{12}+\hat{S}_{12}G_{22}) \\ (G^*_{12}\hat{S}_{11}+G^*_{22}\hat{S}_{21})G_{11} & G^*_{12}(\hat{S}_{11}G_{12}+\hat{S}_{12}G_{22}) + G^*_{22}(\hat{S}_{11}G_{12}+\hat{S}_{22}G_{22}) \end{pmatrix} .$$

Using Lemma 2, the matrix G_{12} can be chosen so that $\hat{S}_{11}G_{12}+\hat{S}_{12}G_{22} = 0$; so that

$$\tilde{S} = \begin{pmatrix} G_{11}^* \hat{S}_{11} G_{11} & 0 \\ 0 & G_{22}^*(\hat{S}_{21}G_{12}+\hat{S}_{22}G_{22}^{-1})G_{22} \end{pmatrix}.$$

Since G_{11} is unitary, we may choose it to diagonalize $\hat{S}_{11}$. This completes the proof of Theorem 2. $\|$

<u>Remark</u>. In the second alternative proof we used the fact that if A and B are positive definite, then both can be diagonalized simultaneously by a nonsingular matrix W, i.e.,

$$A = WW^* , \quad B = WD_\theta W^* , \tag{7}$$

where $D_\theta = \mathrm{diag}(\theta_1,\dots,\theta_n)$, and $\theta_1 \geq \cdots \geq \theta_n \geq 0$ are the characteristic roots of $A^{-1}B$. When the hypotheses of positive definiteness are removed the simultaneous decomposition may no longer be accomplished in general. Theorem 2 complements the above result by providing a simultaneous decomposition. This can be stated in a form parallel to (7).

<u>Theorem 3</u>. If $A \geq 0$, $B \geq 0$, then there exists a nonsingular matrix W such that

$$WAW^* = \begin{pmatrix} I_\tau & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad WBW^* = \begin{pmatrix} D_\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $D_\theta = \mathrm{diag}(\theta_1,\ldots,\theta_\tau)$ are the nonzero characteristic roots of $|B-\theta A| = 0$.

Proof. The result follows from Theorem 2 by noting that if V is a matrix satisfying (5), then for any nonsingular matrix C,

$$W = V\begin{pmatrix} I & 0 \\ 0 & C \end{pmatrix}$$

also satisfies (5). Since $M \geq 0$, there exists a C such that $CMC^* = \mathrm{diag}(I,0)$. ||

An equivalent form of Theorem 3 was obtained (but not published) by Olkin (1951). We suspect that this result is even older, but have not been able to locate a reference.

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